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The History of Paradoxes and Logics

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Abstract

This paper delves into the realm of paradoxes, which encompasses scenarios where confounding conclusions emerge from apparently valid premises or logical reasoning. Paradoxes are classified into two distinct categories: "pseudo-paradoxes," arising due to flawed premises or reasoning that result in contradiction, and "genuine paradoxes," wherein opposing conclusions materialize even though both premises and logic are sound.

Paradoxes go beyond mere puzzles; they trigger deep philosophical and logical inquiries that call for careful consideration and resolution. Throughout history, paradoxes have existed within our cognitive framework, unveiling substantial issues that were previously disregarded, thus leading to moments of crisis. Renowned philosophers underscore the significance of paradoxes. Prominent contemporary analytical philosopher Quine, for instance, highlights how uncovering paradoxes has been pivotal in reshaping the foundations of our thought processes.

Keywords:

What is a Paradox?

The term "paradox" originates from the Greek words "para," meaning "beyond," and "doxa," which translates to "accepted opinion or belief." Therefore, a paradox can be defined as something that goes beyond or contradicts commonly accepted knowledge or beliefs. [1]

R.M. Sainsbury, in his comprehensive work on paradoxes titled "Paradoxes," offers the following definition of a paradox:

"This is what I understand by a paradox: an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises. Appearances have to deceive since the acceptable cannot lead by acceptable steps to the unacceptable. So, generally, we have a choice: either the conclusion is not really unacceptable, or else the starting point, or the reasoning, has some non-obvious flaw." [2]

According to Sainsbury, three conditions must be met for an inference to qualify as a paradox: [3]

1) The premises leading to the conclusion must seem acceptable and devoid of apparent flaws.
2) The reasoning from these premises to the conclusion must seem acceptable and devoid of apparent flaws.
3) In the context of commonly accepted beliefs or views, the conclusion of the inference cannot be accepted.

Illustrating Paradoxes : The Case of the Bald Man

Most paradoxes fit within the framework established by Sainsbury's criteria. To illustrate, let's consider the Paradox of the Bald Man, attributed to the ancient Greek philosopher Eubulides. [4] Originally, this paradox presented a counterintuitive statement: "A person without hair is not considered bald." For simplicity, we'll summarize the paradox's conclusion as: "No matter how many hairs a person has on their head, they can be classified as bald." [5]

Unraveling the Paradox of the Bald Man

The Paradox of the Bald Man can be explained concisely through the following argument [A]: [6,7]

[A]
1. A person with no hair is considered bald.

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2. Regardless of how many hairs an individual has, they are still classified as bald. This is because adding a single hair to a bald person does not change their bald status.

3. Extending this logic, if a person with one hair is bald, then adding another hair, making two hairs, doesn't change their status as bald.

4. Applying this logic further, for any number of hairs 'n', if a person with 'n' hairs is considered bald, then adding 'n+1' hairs also doesn't change their bald status.

Therefore, no matter how many hairs a person has, they are still considered bald according to this argument.

This argument clearly fits Sainsbury's three criteria for a paradox. First, the premises that a person without hair or with any number of hairs is bald seem true and logically consistent, meeting Sainsbury's first criterion. Second, the argument's reasoning, which progressively adds one hair and maintains the person's bald status, is sound, satisfying the second criterion. Finally, the conclusion that a person remains bald regardless of how many hairs they have is counterintuitive and challenges common beliefs about baldness, aligning with the third criterion. Ultimately, this paradox, much like the unexpectedness of Einstein's hair, demonstrates that even with an increasing number of hairs, a person is still classified as bald, leading to a surprising conclusion.

The Bald Man Paradox

Argument [A] demonstrates that the Bald Man Paradox meets the three conditions outlined by Sainsbury. According to Sainsbury[8], this alignment qualifies the deduction leading to the unsettling statement "Regardless of the quantity of hair an individual possesses, they are labeled as bald" as a paradox. However, for a clearer understanding of paradoxes, Sainsbury's definition requires precision. His explanation includes ambiguous expressions like "seemingly acceptable," along with the concept of inference—a cognitive process deducing one proposition from another. [9]

Examining Paradoxes Through Argumentation

Given that the majority of paradoxes follow the structure of deductive arguments, for the sake of simplicity, our focus will be confined to deductive arguments.[10] These arguments can be summarized as follows:

1. If an argument [ ] satisfies the subsequent three conditions, then the argument [ ] is considered a paradox.
2. The premises presented within the argument [ ] are universally acknowledged as true in reality.
3. Argument [ ] is deemed valid.

The conclusion of the argument [ ] is distinctly identified as false when compared to accepted beliefs or perspectives, such as common sense or established knowledge.

In the context of deductive arguments, its premises must be factually accurate and valid for an argument to be sound. Furthermore, if an argument is valid and all its premises are true, then the possibility of the conclusion being untrue is eliminated; thus, the conclusion of a sound argument must inherently be accurate. Therefore, according to point 3), paradoxes can be explained through the following three scenarios:[11]

1) All premises are acknowledged as objectively true, yet some may not hold true in reality.

2) The argument appears valid, but its validity is questionable upon closer examination.

3) Upon considering embraced beliefs or viewpoints (such as common sense or established knowledge), the argument's conclusion is evidently false ; nevertheless, from a logical standpoint, the conclusion must be acceptedwithout dismissing certain premises.

These scenarios highlight the complexity of paradoxes, challenging our understanding of truth and validity within deductive arguments.

Exploring Paradoxes

Many paradoxes fall into a specific category, with notable examples including Zeno's Arrow Paradox, [12] the Barber Paradox, [13] and Bertrand's Box Paradox. [14] However, paradoxes in this category often arise from the incorrect assumption that invalid arguments are valid. Strictly speaking, those in the second category aren't genuine paradoxes but rather logical fallacies.

The majority of significant paradoxes fall into either case 1 or case 3. Case 1 involves statements that we believe to be true but, upon closer examination, are found to be false. Conversely, case 3 involves conclusions that contradict our everyday intuition or established knowledge, yet we are compelled to accept them without refuting the underlying premises. As observed earlier, the Paradox of the Bald Man falls into case 1.

Achilles and the Tortoise Paradox[15,16]

Within this paradox, Achilles and the Tortoise engage in a race, with the Tortoise being given a head start. If the Tortoise starts ahead of Achilles, then despite Achilles being faster than the Tortoise, he can never catch up. By the time Achilles reaches the point where the Tortoise started, the Tortoise will have moved a certain distance ahead, and this process will continue endlessly.

Fletcher's Paradox (Paradox of the Arrow)[17,18]

An illustrative example falling into the second category is Zeno's Paradox of the Arrow. According to this paradox, as the arrow moves through the air, it occupies every point it passes through at any given moment. Since time cannot be broken down into separate instants, the arrow appears to remain still at each moment, despite its overall motion. Thus, the concept of the arrow in flight seems to conflict with its actual movement. While the Paradox of Achilles and the Tortoise deals with spatial division, the Paradox of the Arrow delves into the concept of time.

Is the Whole Greater than the Part?[19]

The concept of a "set" holds significant importance in mathematics, being both fundamental and essential. Sets find extensive application across various mathematical disciplines, and the establishment of set theory in the late 19th century fundamentally shaped mathematics. In this discussion, we will explore the challenges associated with essential mathematical concepts such as "infinity" and "sets."

Is the set of natural numbers bigger than the set of even numbers?

Mathematical sets are divided into two main types: finite sets, with a specific number of elements, and infinite sets. For finite sets, we can count their elements by establishing a one-to-one correspondence with a subset of natural numbers, effectively measuring their size. But how does this apply to infinite sets? Can we compare the sizes of infinite sets using the concept of cardinality? The answer is yes.

During the 1870s, the mathematician Georg Cantor developed the concept of "one-to-one correspondence" and formulated methods for calculating the cardinality of infinite sets, enabling the comparison of their sizes.[20]Cantor's groundbreaking work laid the foundation for modern mathematics, particularly in how infinity is quantified and understood within modern algebra. Yet, applying Cantor's methods to quantify the infinite can lead to contradictions with our everyday intuition, sometimes resulting in paradoxical situations.

The Infinite Hotel Paradox (Hilbert's Paradox of Infinity)[21,22]

To illustrate an extreme example falling into the second category, we can examine Hilbert's Paradox of the Infinite Hotel.

On a certain day, an infinite number of guests were lodged within an infinite hotel, and there were no available rooms. At that moment, a new guest arrived. The hotel owner devised a clever solution: He requested each guest to move to a room with a number one greater than their current room number. Consequently, the guest in Room 1 moved to Room 2, the guest in Room 2 moved to Room 3, and so forth. Because the hotel has an infinite number of rooms, Room 1 became vacant, allowing the new guest to check in. Nevertheless, on the following day, an infinite number of new guests arrived at the hotel. Subsequently, the hotel owner requested all the existing guests to shift to rooms with numbers twice the value of their original room numbers. The occupant of Room 1 moved to Room 2, the guest in Room 2 relocated to Room 4, the occupant in Room 3 moved to Room 6, and this sequence continued. Through this process, the current guests transitioned to rooms with even numbers, creating vacancies in all rooms with odd numbers for the new arriving guests to occupy.

From a mathematical perspective, this narrative holds true. It generates a paradoxical circumstance due to properties that challenge our understanding of infinity, thereby engendering a comparable paradox.

Within the finite realm, the collection of odd numbers is of lesser cardinality than the set of natural numbers. Yet, in the infinite realm, a direct and complete one-to-one correspondence between all even numbers and all natural numbers does exist, resulting in the two sets being of equal size. The intriguing trait of infinite sets lies in the fact that a subset can possess the identical size as the original set.

This paradox can be outlined as follows:

1. The set of natural numbers is a subset of the set of integers.

2. While the set of natural numbers extends infinitely in one direction, the set of integers extends infinitely in two directions.

Natural numbers = {0, 1, 2, 3, ...} Integers = {...-3, -2, -1, 0, 1, 2, 3, ...}

3. A direct one-to-one correspondence exists between the set of natural numbers and the set of integers.

4. In cases where a one-to-one correspondence exists between two sets, these sets share the same size.

Assuming all the premises are true, the conclusion must consequently be true, rendering the argument valid. Furthermore, premises 1, 2, and 4 are evidently true. Hence, if we can establish the truth of premise 3 as well, the argument is sound. This paradox falls under the category of the third type of paradox.

Liar's Paradox[23,24,25,26]

The liar's paradox is a classic example of a self-referential paradox that involves a statement referring to itself. One prominent example is the statement:

"This statement is false."

This sentence, in essence, represents a slight modification of Epimenides' statement. However, it raises the question : If you assume that the statement is true, then it must be false as it claims to be false. However, if you assume that the statement is false, then it must be true since it's claiming to be false but isn't, creating a contradiction.

The liar's paradox highlights the challenges and complexities that arise when dealing with self-referential statements and the potential for logical contradictions. It's a well-known example of a paradox within philosophy and logic, and it has inspired discussions about the limits of language, self-reference, and the nature of truth in formal systems.Various approaches and theories have been proposed to handle or resolve the liar's paradox, contributing to the ongoing exploration of the foundations of logic and mathematics. Here are some of the key solutions that have been proposed:

Tarski's Semantic Hierarchy:[27,28]Alfred Tarski introduced a hierarchical model that distinguishes between object languages and meta-languages, aiming to define truth in a manner that circumvents direct self-reference. This approach separates the language being discussed (object language) from the language used to discuss it (meta-language), thus providing a structured way to talk about truth without falling into paradoxes.

Paraconsistent Logic :[29] Some philosophers and logicians have explored paraconsistent logics, which tolerate contradictions to a certain degree. These logics are designed to handle situations where a statement and its negation can both be true without leading to inconsistency.

Dialetheism :[30] Dialetheism is the view that there can be true contradictions. Some proponents of this view argue that the liar's sentence "This statement is false" can be both true and false simultaneously.

Revision Theory :[31] This approach, developed by mathematician Kripke[32] and philosopher Field,[33] suggests that certain paradoxical sentences should be "revised" tocircumvent self-reference and paradox.This method aims at reinterpreting or modifying statements that lead to contradictions, offering a unique solution to dealing with logical paradoxes.

Restrictions on Self-Reference :[34] Some solutions involve imposing restrictions on the application of self-reference in language or formal systems. For instance, systems could be structured to prevent circular reference. This preventive measure helps maintain logical consistency within the system.

Formal Axiomatic Systems :[35] Various formal systems, like Russell's Type Theory[36] or Zermelo-Fraenkel set theory[37] with the Axiom of Regularity, have been developed to address self-referential paradoxes.They achieve this by introducing rules or axioms that deter the formation of paradoxical statements.By establishing clear boundaries for construction within the system, these axiomatic frameworks aim to maintain logical consistency and avoid contradictions.

Incompleteness: The liar's paradox serves to underscore the inherent limitations of formal systems, particularly highlighting the undecidability of certain propositions within these frameworks. This concept is further evidenced by Gödel's incompleteness theorems[38], which assert that within any sufficiently complex formal system, there are propositions that cannot be proven or disproven using the axioms within that system. This revelation points to the intrinsic limitations and complexities of formal logical systems.

It is important to understand that there is no single, universally accepted solution to the liar's paradox. Different solutions have their own merits and drawbacks, often rooted in distinct philosophical principles. The paradox remains a subject of vigorous debate, with philosophers, logicians, and mathematicians continually exploring new insights and approaches. While numerous scholars have proposed solutions, let's take a closer look at Russell and Tarski's solutions more in detail.

Bertrand Russell identified significant issues arising when the "vicious circle" principle is violated, leading to confusion and paradoxes. To address this, he introduced a systematic solution known as the "Theory of Types." Specifically, Russell proposed the "Theory of Ramified Types"[39] as a method to resolve paradoxes such as the liar paradox (e.g., "This statement is false"). The core objective of the Ramified Type Theory is to categorize paradoxical statements into hierarchical levels, or "types." Statements are assigned to a higher type that references statements of a lower type, but crucially, they do not refer to themselves directly. The Theory of Ramified Types, as a methodology, organizes propositions into a structured hierarchy, classifying sentences based on their content and subject matter into distinct categories. In this framework, first-order propositions encompass self-contained sentences that either standalone without requiring additional information or focus on particular subjects.

These propositions are straightforward, dealing with basic or specific topics. On the other hand, second-order propositions refer to sentences that discuss or analyze first-order propositions. These are essentially comments or evaluations of the sentences in the first-order category, exclusively addressing the content or structure of those initial propositions. Overall, Russell's ramified type theory is an attempt to establish a structured and orderly framework for reasoning and formal language that avoids the pitfalls of self-reference and paradoxes. It represents a significant advancement in the fields of formal logic and the philosophy of mathematics.

Alfred Tarski, a renowned Polish-American mathematician and logician, introduced a solution to the liar's paradox known as the "Tarski Hierarchy" or the "Semantic Hierarchy." [40,41]This approach is to solve self-referential paradoxes, such as the liar's paradox, by clearly distinguishing between two levels of language: the language of the theory itself and the meta-language used to describe or analyze that theory. Tarski's solution emphasizes the importance of separating the discussion about a theory (meta-language) from the theory's own language (object language). Through this hierarchical separation, Tarski's Semantic Hierarchy provides a framework for understanding and resolving paradoxes inherent in languages that discuss themselves.

Tarski's solution can be understood in the following steps:

Tarski's Object Language and Meta-language Distinction:Tarski highlighted the difference between the language being talked about (e.g., English) and the language used to discuss it.

Truth Definition: Tarski introduced a formal definition of truth for sentences in the object language. He achieved this by using another language (meta-language) to describe how we decide if sentences are true. This definition is both recursive, allowing it to apply to sentences of varying complexity, and hierarchical, establishing a structured way to assign truth values. This systematic approach enables a clear and paradox-free understanding of truth in language, distinguishing between sentences based on their complexity and the language level they belong to.

Hierarchy of Languages: Tarski introduced a hierarchy of languages, each referring to the one below it but not to itself. This arrangement prevents self-reference and the emergence of paradoxes. In the liar's paradox, the sentence "This statement is false" would belong to a higher-level language discussing lower-level statements. As this higher-level sentence doesn't directly reference itself, the paradox is avoided.

Tarski's approach does not completely eliminate self-reference; rather, it carefully regulates its usage to avoid paradoxes. By distinguishing between the primary language (object language) and the language discussing it (meta-language), and by implementing a hierarchy of languages, Tarski aimed to provide a formal framework in which truth could be defined without leading to self-referential paradoxes. It's important to note that Tarski's solution represents just one of several proposed approaches to addressing the liar's paradox and similar logical challenges.

Set Theory Paradox[42]

According to Cantor's interpretation, a set is described as "a collection M of well-determined and well-distinguished objects within our intuition or thought, considered as a whole (in this context, these objects are referred to as 'elements' of M)." This definition of sets, proposed by Cantor, is commonly known as the "naive conception of set." However, this straightforward interpretation of sets was later replaced by alternative axiomatic systems, prompted by Russell's discovery of the set theory paradox around 1901. [43]

Russell's paradox is a famous example in set theory that challenges the foundations of mathematics. Consider the set of all sets that do not contain themselves as elements. Now, ask the question: Does this set belong to itself as an element or not? If we assume that the set does belong to itself, then it must satisfy the condition of not containing itself, which leads to a contradiction. Conversely, if we assume that the set does not belong to itself, then it meets the condition of sets that do not contain themselves, which again leads to a contradiction. This paradox exposes a fundamental inconsistency within the naive set theory and led the development of more rigorous axiomatic systems for set theory.

To illustrate the concepts discussed above, let's examine the following example. Russell's librarian story[44]presents a self-referential statement that gives rise to a paradox within set theory. It is closely related to Russell's Paradox, which is a fundamental paradox in the foundations of mathematics and set theory. To better organize the library's books, the librarian created two lists. One list showed books and their own list titles, while the other list only showed books without referencing its own title. But the librarian faced a problem. Should the second list, which has books that don't say their own names, include itself? If it does, it goes against what the librarian wanted to do, because it should only have lists that don't introduce themselves. Conversely, if it doesn't include itself, then by definition, it should be on the list. This put the librarian in a confusing situation with no clear answer.The list in the librarian's story is a bit like the idea of a set that Frege used to explain numbers. Russell's solution to this issue was to make a new rule that stops sets from having themselves as part of the set. This helps prevent the confusing situation we talked about. In 1910, Russell and his colleague Alfred North Whitehead released the first volume of Principia Mathematica.[45] In this book, they made progress in solving the tricky problem Russell had made Russell's Paradox and related set-theoretic paradoxes played a significant role in shaping modern set theory and contributed to refining foundational principles in mathematics.

Russell's Barber Paradox[46,47]

Russell's Barber Paradox stands out as a unique paradox, separate from those of set theory or the liar's paradox. Russell introduced it to highlight the difficulties with self-referential statements and the inconsistencies they can produce.

Russell's paradox is based on examples like this. In a particular village, there is a barber who exclusively shaves only those individuals who do not shave themselves. The question is: does the barber shave his own or not? If he shaves himself, he becomes someone who shaves his own, contradicting the condition. Therefore, he should not shave himself. Conversely, if he does not shave himself, he falls into the category of people for whom he should shave those men, again contradicting the condition. Hence, he would be a man who does shave men who shave themselves.

While sharing similarities with self-referential paradoxes such as the liar's paradox, the Barber Paradox doesn't neatly align with either the set theory or the liar's paradox categories. Instead, it stands as a unique demonstration of logical intricacy. Russell himself encountered a paradox that deeply challenged the foundations of mathematics and logic, famously known as "Russell's paradox." This paradox arises when considering "all sets that do not include themselves as elements." The question arises: does this set include itself as an element or not? Assuming it does, we reach the conclusion that it does not, and assuming it does not, we arrive at the conclusion that it does.If we assume it does, we are led to conclude it does not; conversely, if we assume it doesn't, we are drawn to the conclusion that it does.

No set can belong to all sets that exclude itself. This follows from the following reasoning: if such a set existed and it included itself, then it would only belong to sets that do not include themselves. Conversely, if the set didn't include itself, it would be an element of all sets that don't include themselves, paradoxically including itself. Therefore, the necessary and sufficient criterion for a set to be one of its own members is that it isn't a member of itself. This paradox leads to the conclusion that no set can simultaneously belong to all sets without self-inclusion while exclusively comprising such sets.

Gottlob Frege[48]

Among the renowned logicians, Gottlob Frege stands out as a contemporary of Cantor, sharing a parallel period. However, during his time, Frege's recognition remained limited due to the intricacy of the symbols he employed. Gottlob Frege made a significant contribution to the birth of new logic and can be regarded as the pioneer of symbolic logic. Frege's work laid the foundation for modern symbolic logic, introducing a new system that transformed the analysis of quantified statements and standardized the concept of "proof" in logic. He demonstrated that when we use logical systems, complex mathematical ideas can become much clearer and more precise when we talk about them using logical and mathematical language. This groundbreaking approach profoundly revolutionized the classical logical framework, which had seen little advancement since the days of Aristotle over two thousand years ago.[49]

Frege aimed to derive important elements of mathematics from "logic." For instance, he attempted to derive the axioms of number theory from logic. Unfortunately, his system was eventually found to be inconsistent, and his lifelong aspiration to "reduce mathematics to logic" was not realized. Nevertheless, Frege's new ideas in logic paved the way for new possibilities and continue to exert a notable influence to this day.

In 1893, Frege published his significant work, "The Basic Laws of Arithmetic (Die Grundgesetze der Arithmetik)." However, in 1902, as he was preparing the second volume of "Grundgesetze, he received a letter from Bertrand Russell pointing out a paradox. Realizing the significance of this paradoxical dilemma, Frege engaged in an extensive correspondence with Russell. Ultimately, he addressed the paradox in the appendix of the second volume, published in 1912, using the framework of his own system.[50]

The impact of Russell's paradox on Frege's logical system is quite dramatic, leading some to mistakenly believe that his entire effort was in vain. Yet, Frege's contributions remain important. When he initially introduced his logical system, it didn't garner much attention from mathematicians. However, following the early 20th century, Frege became famous, and many young mathematicians, like Gödel, became deeply engaged in the study of his innovative logic. From today's perspective, Frege's contributions can be summarized into two key points. Firstly, he led the way in using formal symbols to express the relationships between propositions or statements within the framework of symbolic logic. Secondly, he tried to figure out how number systems are constructed and the overall structure of mathematics using "logic." In essence, Frege began the process of translating mathematics into logical terms, ultimately resulting in the convergence of logic and mathematics. Frege introduced his logical symbols while developing the predicate logic system. [51] For example, consider the statement "Aiden Hyun is a human." In this system, "Aiden Hyun" is represented by the symbol A (alpha), and "is a human" is symbolized by H. Thus, the statement can be succinctly expressed as "HA."

Gödel's Incompleteness Theorems and the Fall of Formalism

Hilbert's vision of creating a flawless logical system where mathematics could be entirely represented through symbols and rules appeared to be accomplished. Not only Hilbert himself but also many mathematicians of his time believed that while not immediate, this goal would eventually be realized. However, the basis of Hilbert's envisioned formalist mathematics collapsed when the young Austrian mathematician Kurt Gödel introduced the Incompleteness Theorems in 1931.[52]

The "Hilbert Program" aimed to achieve the ultimate goal of demonstrating the consistency of all mathematical theories. This would establish a foundation of "absolute truth," ensuring the coherence of all of mathematics.

Gödel's Incompleteness Theorems consist of two key propositions:[53]

First Theorem: In any consistent arithmetic axiom system, there are true statements that cannot be proven. Essentially, this means the system is incomplete. Within any theoretical system based on axioms without contradictions, there will always be theorems that cannot be proved or disproved.

Second Theorem: In any arithmetic axiom system that is sound (free of contradictions), it is impossible to deduce from the system itself that the system is consistent. There's no way to validate the accuracy of a theory solely based on its own axioms. It simply shows the system cannot demonstrate its own consistency.

The first theorem highlights a fundamental aspect of mathematics: no matter the chosen set of axioms, there will always be questions that remain unanswered, underscoring the inherent incompleteness of mathematics. Furthermore, the second theorem goes a step further by asserting that mathematics itself cannot guarantee that the chosen axioms do not lead to contradictions. This effectively establishes that there's no way to demonstrate that a mathematical system is free of contradictions. Ultimately, Gödel's work demonstrated the impossibility of achieving Hilbert's program.

Let's express Gödel's theorem mathematically as follows. Gödel's proof of the incompleteness theorem relies on two fundamental concepts. Firstly, every mathematical proposition is translated into numbers, specifically natural numbers. In simpler terms, each proposition is associated with a natural number, representing the proposition itself (allowing us to identify the proposition through its corresponding number). This assigned number is called the Gödel number. Secondly, using a somewhat complex recursive method in conjunction with Gödel numbers, we identify a unique statement, referred to as proposition G, as follows:

Proposition G : "G is unprovable."

Similar to Russell's paradox, we encounter a paradoxical situation here. Proposition G claims it cannot be proven. However, if we assume G can be proven, we arrive at a contradiction where proving G validates its unprovability, making it both unprovable and irrefutable If this system maintains its consistency, then proposition G is valid, yet its proof remains nonexistent, rendering the system incomplete. This idea forms the basis of the first theorem's proof, and understanding this concept makes proving the second theorem much simpler. [54]

Gödel's incompleteness theorems are not paradoxes in the traditional sense but rather profound mathematical results in the philosophy of mathematics, demonstrating that there are limits to what can be achieved through formal mathematical systems. These theorems are based on solid mathematical principles, grounded in formal logic and set theory, and are free from self-contradiction or illogical elements. However, they question some of the fundamental assumptions about mathematics and the limitations of formal systems.

Just as Russell's paradox was illustrated through the story of a librarian, Gödel's first incompleteness theorem can be clarified using the "liar's paradox," also known as the "Cretan Liar Paradox."[55]

In Crete, a man named Epimenides once made a simple and clear statement one day:

"I am a liar!"

When attempting to determine whether this statement is true or false, it immediately leads to a paradoxical outcome. Let's start by assuming that the statement is true. Then, Epimenides is a liar. However, if we assumehis statement to be true, it logically follows that he must be speaking the truth. Hence, he cannot be a liar. This contradiction arises regardless of whether the statement is true or false. Now, let's assume that the statement is false. In this case, Epimenides is not a liar. But based on our assumption of the statement's falsity, he must be lying, which means he is indeed a liar. Once again, we find ourselves in a contradictory situation. Consequently, regardless of whether we consider Epimenides's statement true or false, it consistently yields a paradoxical result. Therefore, this statement cannot be categorized as either true or false.

Gödel reinterpreted Epimenides' liar paradox introducing the concept of "proof" to construct his own paradox.[56]He proposed a statement:

"This statement contains no proof."

If we consider this statement false, then there must exist a proof establishing its falsehood, which directly contradicts the statement's own assertion. To avoid contradiction, we must then accept the statement as true. Yet, even if it is true, the statement itself remains unprovable, consistent with its content. By translating the preceding statement into mathematical language, Gödel demonstrated the existence of mathematical propositions that are true yet cannot be proven; these are termed undecidable propositions within mathematics. This discovery significantly challenged Hilbert's program, which aimed at establishing a complete and consistent foundation for all of mathematics.

The definitions of "completeness" and "consistency" were previously explained in the context of Hilbert's formalism. Gödel's incompleteness theorem, however, introduces a sophisticated idea that may not be immediately understandable to everyone. Readers can grasp it at a foundational level, for instance, by understanding that "a complete arithmetic system is nonexistent." This basic level of comprehension is enough for a foundational understanding of the theorem.

Four years before Gödel introduced his incompleteness theorem, physicist Werner Heisenberg proposed the theory of "uncertainty principle." This uncertainty principle highlights that, just as there are fundamental limits to proving mathematical theorems, there are also natural limits to how precisely we can measure physical properties. For example, when trying to accurately measure the position of an object, one must inevitably sacrifice the precision of measuring its velocity. This principle reveals the innate uncertainty present in the physical world.

To measure the position of an object, we use light. However, light consists of energy-carrying particles called photons. For a precise measurement, we need to shine a lot of photons, carrying a significant amount of energy, onto the object. This bombardment of photons alters the object's original velocity due to the collisions. As a result, the more precisely an object's position is determined, the more uncertain its velocity becomes. This means it's impossible to achieve absolute precision in measuring both an object's position and velocity at the same time. This fundamental idea is what's famously known as the "uncertainty principle." [57]

Heisenberg's Uncertainty Principle, a cornerstone of quantum physics, establishes a limit on how precisely we can measure physical properties. It's not a paradox because it's grounded in well-established mathematical and experimental foundations. Yet, its revolutionary impact on both mathematics and physics introduced a profound idea: "there is no absolute and certain truth; only uncertain and relative truths exist." This shift in understanding has significantly influenced other fields, such as philosophy, linguistics, and economics, marking a philosophical turning point across disciplines.

Neither True Nor False : Three-Valued Logic[58]

(A) "This statement is false."

Prominent philosophers like D.A. Bochvar and B. Skyrms advocate for introducing a third truth value, labeled as 'paradoxical,' for statements involved in the liar's paradox due to their inherently paradoxical nature. [59] They argue that because of the paradoxical outcomes arising from statements like (A) within the liar's paradox, employing three-valued logic becomes necessary. This approach assigns a truth value of 'paradoxical'—neither true nor false—to statements entangled in the complexities of the liar's paradox. [60,61] However, fundamentally, this proposition cannot be seen as a rational and successful attempt to resolve the liar's paradox.

This proposition challenges the foundational principle of two-valued logic, which is essential to propositional logic, by going against the common-sense notion that a statement must be either true or false. This approach essentially argues that our traditional way of assessing truth and falsehood doesn't apply in cases like the liar's paradox. It doesn't offer a direct solution to the liar's problem. Instead, it suggests that due to the peculiar outcomes generated by sentences like (A) in the liar's paradox, these sentences shouldn't be strictly categorized as true or false. Rather, they are labeled as 'paradoxical' sentences. And if we agree with this idea, it creates a new version of the liar's paradox.

Consider the statement (C):

"This statement is false or paradoxical." (1)

When applying a three-valued logic perspective to (C), we encounter the paradoxical conclusion that if (C) is true, then it must be either false or paradoxical. Similarly, if (C) is false or paradoxical, it must be true.(4)

1. According to their argument, (C) can hold the truth value of true, false, or paradoxical.

2. If we interpret (C) as true, then it implies that (C) must be either false or paradoxical (since (C) claims to be false).

3. On the other hand, if we assume (C) is false, it directly leads to the conclusion that (C) is true.

4. Similarly, if (C) is paradoxical, it still leads to the conclusion that (C) is true.

5. Considering points 2, 3, and 4, we can deduce that if (C) is true, it results in the scenario where (C) must be either false or paradoxical. Conversely, if (C) is false or paradoxical, it leads back to the conclusion that (C) is true. [62]

Neither True Nor False : Absence of Truth Value[63]

Philosophers such as Bar-Hillel, N. Garver, and W.C. Kneale argue that statements involved in the liar's paradox cannot be considered genuine propositions or statements. They argue that such statements do not qualify for a truth value of either true or false because, in their perspective, only genuine propositions or statements can hold truth values. Consequently, they suggest that statements like those found in the liar's paradox lack truth values altogether and should be seen as neither true nor false. Saul Kripke echoes a similar viewpoint, introducing the notion of "groundedness" to support this argument.

Types of Paradoxes in 20th Century

Throughout the 20th century, scholars have been refining how paradoxes are categorized, developing more intricate criteria for their classification. Two key concepts in this discussion are contradiction and infinite regress.

A contradiction occurs when two opposing statements are both claimed to be true simultaneously, such as "A is true and B is true," when A and B are fundamentally opposite. This just doesn't work because these ideas can't both be true at the same time. Sometimes, even with only one idea, contradictions can emerge, as we've observed.[65]

Infinite regress, on the other hand, happens when two ideas that might individually seem logical are placed together, leading to an unexpected outcome—they negate each other and result in a paradox. This creates a situation where, instead of complementing each other, the ideas perpetuate a cycle that doesn't resolve, highlighting the complexity and intrigue of paradoxes. [66]

What distinguishes contradiction from infinite regress is the presence of self-reference. In contradiction, the two ideas refer to themselves in a manner that prevents them from both being true. On the other hand, infinite regress avoids this self-reference and instead points to a different idea. The examples provided below illustrate the concepts of the contradiction and Infinite Regress Paradox.

An example of a contradiction paradox is the statement: "This statement is false." If we consider the statement to be true, then it must be false, as it declares itself not true. However, if it is false, this would imply it is true. This creates a contradiction where we cannot definitively categorize the statement as true or false.

To illustrate the Infinite Regress Paradox, consider a situation where you need a key to open a box, but the key is inside the box itself. To get the key, you must open the box, but you cannot open the box without the key. This scenario creates an infinite loop: you can't access the box without the key, and you can't obtain the key without opening the box. The process never concludes, resulting in an infinite regress of dependencies. These examples demonstrate how contradictions and infinite regress paradoxes can challenge our logical understanding and lead to paradoxical situations.

Self-Reference

A famous example of self-reference is Epimenides' paradox: "All Cretans are liars." This is commonly known as the "liar paradox." Self-reference occurs when a subject or statement refers to itself, and in this case, it involves a "self-containing set," a concept not allowed in modern set theory. The paradox arises because the statement contradicts itself by negating its own validity, leading to a logical contradiction. [67]

As mentioned earlier, a significant trait of paradoxical statements such as "This statement is false" or "Consider the set of all sets that do not contain themselves. Does this set contain itself?" are their self-referential nature. The statement "This statement is unprovable," emphasized in Gödel's incompleteness theorem, inherently refers to itself much like the assertion of the liar.

Willard Van Orman Quine, a mathematician and philosopher, explores paradoxes in his work and demonstrates that paradoxical sentences aren't limited to those that are self-referential. He points out that even sentences lacking indexicals or demonstratives—words such as "this" or "that"—can still embody paradoxes. Quine's paradox can be summarized as follows: "Yields falsehood when preceded by its quotation." [68]

The statement "Yields falsehood when preceded by its quotation" becomes false when its own quotation precedes it. Here's a step-by-step breakdown to clarify the paradox:

a) "It" refers to the statement "Yields falsehood when preceded by its quotation."

b) "Its quotation" means the phrase "Yields falsehood when preceded by its quotation."

Thus, placing "it" before "its quotation" results in: "'Yields falsehood when preceded by its quotation' yields falsehood when preceded by its quotation." [69]

Reflecting on these components allows us to examine the paradox more thoroughly. It suggests that the sentence "'Yields falsehood when preceded by its quotation' yields falsehood when preceded by its quotation" is not truthful. Essentially, the sentence declares it is not accurate, leading to a complex scenario. If the sentence is indeed false, then according to its own claim, it should be true, thus generating a contradiction. [70]Furthermore, Willard Quine identified three categories of paradoxes beyond self-reference: veridical paradoxes, which reveal surprising truths; falsidical paradoxes, which are seemingly true statements that actually are false; and antinomies, which are contradictions arising from a set of apparently valid principles. [71]

Veridical Paradox – Unexpected Conclusion

A veridical paradox is a type of paradox in which a seemingly absurd or contradictory situation is actually true. In simpler terms, the outcome of the paradox, though counterintuitive, is indeed factual. The truth paradox refers to situations where both the process of argumentation and the conclusion are correct, but the conclusion feels paradoxical because it's significantly different from what we anticipated. A classic example of the truth paradox is the birthday paradox. [73]

The Birthday Paradox poses an intriguing question: How many individuals do you think need to be in a random group for there to be a 50 percent chance that at least two of them share a birthday?

The birthday paradox is about the likelihood of having at least two people with the same birthday in a group of n people selected at random. Though it is not technically a paradox, it is often referred to as such because the probability turns out to be surprisingly high. When people quickly think about it, many would assume that you'd need around 182 people, which is roughly half the number of days in a year. But would you actually need 182 people in a group to find a birthday match?

The true answer is surprisingly low: just 23 people. [74]

The reason why the answer of 23 seems unexpected is because the birthday paradox involves mathematical calculations using exponents, which consider the compounded effects of pairing each person in the group with every other person. This is something our brains don't naturally account for when estimating probabilities, leading to the paradox's seemingly puzzling result.

A fascinating example of a truth paradox in the history of mathematics is the 'Banach-Tarski Paradox.' This paradox claims that it's possible to divide a cake into several pieces and, by simply rotating and rearranging these pieces, miraculously produce two cakes of the same size as the original. Remarkably, this paradox becomes reality in the context of objects made up of infinitely many points, showcasing a peculiar aspect of theoretical mathematics. The validity of the Banach-Tarski Paradox relies on something called the 'Axiom of

Choice' in the current mathematical framework.[75]This axiom enables us to select an infinite number of items from a set with numerous elements. However, this Axiom of Choice has given rise to various paradoxes and sparked debates about whether it should be a part of mathematics. This discussion remains one of the most significant controversies in mathematical history.

This paradox involves a counterintuitive mathematical theorem related to the division and rearrangement of sets in three-dimensional space. It delves into the strange behaviors of specific mathematical entities called "paradoxical decompositions." The Banach-Tarski Paradox is often utilized to illustrate the odd and puzzling aspects of certain mathematical concepts. It highlights what might happen when these concepts are taken beyond math and applied to the real world, even if the results do not quite align our usual understanding of physical objects and space.

Examples like Hilbert's Paradox of the Grand Hotel[76] and Schrödinger's Cat[77] also serve as illustrations of this paradox.

A Falsidical Paradox: Deceptive Conclusions

The falsehood paradox occurs when a illogical conclusion is reached due to an error in the process of argumentation. A well-known example of this is the '1 = 2' proof.[78]

$a = b$

$a^2 = ab$ (multiply by 'a' both sides)

$a^2 – b^2 = ab – b^2$ (subtract b^2 from both sides)

$(a + b)(a – b) = b(a-b)$ (factoring)

$a + b = b$ (divide by 'a-b' both sides)

$2b = b$ (since $a = b$)

$2 = 1$ (divide by 'b' both sides)

The argumentation might appear flawless at first glance, but upon closer examination, dividing the factored equation by (a-b) in the fifth line equates to dividing by zero, which is an invalid operation. This process is similar to dividing both sides by zero, resulting in the erroneous conclusion that 2 equals 1, rendering the argument faulty.

Zeno's paradoxes are instances 'falsidical' reasoning. These paradoxes propose scenarios such as a flying arrow never reaching its destination or a swift runner being unable to catch up to a tortoise that starts with a slight lead. [79,80]

Antinomy: Violation of Intuition Paradox[81]

Antinomy refers to a self-contradictory paradox that uses accepted methods of reasoning to reach a conclusion that contradicts itself. An example of this is the Grelling–Nelson paradox,[82,83] which highlights real challenges in our understanding of concepts like truth and description.In the case of "Grelling and Nelson's paradox," it categorizes adjectives into two groups: "autological adjectives" and "heterological adjectives." However, the term "heterological" itself falls into the category it defines, making it an autological adjective. Consequently, the term "heterological" cannot be included in the set of meanings it's intended to represent. [84]

Quine classified paradoxes that result in self-contradiction as "antinomies." [85] Among these, the "violation of intuition" paradox is particularly intricate and profound. It emerges when the process of argumentation is valid, yet the conclusion contradicts our logical intuition. The "liar paradox" is a key example of this type.

'This sentence is false.'

Let's assume this sentence is true. Then, according to its content, it must be false, leading to a contradiction. Now, suppose the sentence is false. In that case, the negation of the sentence, 'This sentence is true,' holds true. Consequently, the sentence must be true. This loop of contradictions exemplifies the liar paradox.

The self-reference paradox and the antinomy paradox may seem similar, but they are actually distinct concepts. The self-reference paradox occurs when a statement refers to itself in a way that generates a contradiction or a logical issue. [86] An example is the well-known "liar paradox," where a statement declares itself to be false. On the other hand, antinomy refers to a type of paradox where contradictory principles or ideas coexist within the same system or framework. It arises when valid arguments lead to opposing conclusions. An example is Zeno's paradox of Achilles and the Tortoise, where infinite divisions result in conflicting interpretations of motion and reaching a destination.

In summary, although both self-reference and antinomy involve paradoxes, they are characterized by distinct features and result in different types of logical challenges.

Classic Logic vs Modern Logic

Classical logic differs from modern logic in two significant ways. First, classical logic uses everyday language to express logical arguments, whereas modern logic employs formal symbols, similar to those in mathematics, for the manipulation of logical concepts. Second, while Aristotelian (classical) logic considers both the intension (the inherent qualities) and the extension (the range of application) of a concept, modern logic focuses solely on the extension. This shift from classical logic's emphasis on the 'subject' of a statement to modern logic's focus on the 'statement' itself represents a fundamental change. This transition was enabled by the pioneering work of Cantor and Frege, who expanded upon Leibniz's initial groundwork.

The key difference between classical and modern logic lies in their fundamental units of logic: classical logic focuses on the 'subject,' whereas modern logic prioritizes the 'proposition.' For instance, consider the conditional statement 'If p, then q.' Classical logic might interpret this as 'If it's a white horse, then it's a horse,' focusing on the subjects involved. In contrast, modern logic interprets it as 'If we say that the thing is a white horse, then we also imply it's an animal,' focusing on the propositions made about the subjects. Here, classical logic treats 'p' as a phrase, such as "white horse," while modern logic treats it as a full sentence, emphasizing the implications of stating "the thing is a white horse." [87] This evolution from subject-based to proposition-based logic marks a significant shift and aligns closely with semantics, which explores the meanings behind statements, propositions, and their relation to the world.

Studies on Paradoxes in various fields.

Studies on paradoxes have played a significant role across various fields, from modern physics to the social sciences. These perplexing and counterintuitive situations often challenge our understanding of reality and logic, leading to groundbreaking insights and innovative ways of thinking. Below, we explore how paradoxes have influenced different areas of study:

Modern Physics:

Schrödinger's Cat Paradox:[88] This famous thought experiment in quantum mechanics involves a cat placed in a box with a radioactive atom that may decay, releasing poison. Until observed, the cat theoretically exists in both alive and dead states simultaneously, illustrating the bizarre implications of quantum superposition and the role of observation.

Twin Paradox:[89] In relativity, one twin travels in a spacecraft while the other stays on Earth. The traveling twin returns younger due to time dilation effects. This paradox challenges our conventional understanding of time and space by highlighting the relativistic effects on observers in motion.

Math and Probability:

Berry Paradox :[90]This paradox emerges from the definition of "the smallest number not describable in under twenty words." This definition paradoxically leads to a contradiction, as the phrase itself is a description within the limit, highlighting the complexities and limitations of self-reference and language in mathematics.

Banach-Tarski Paradox:[91] In the field of geometry, this paradox asserts that a solid sphere can be decomposed into a finite number of disjoint subsets, which can then be rearranged through rigid motions to form two identical copies of the original sphere. This counterintuitive result relies on the principles of infinite sets and non-Euclidean geometry, challenging our understanding of space and volume.

Social Studies:

Paradox of Tolerance:[92]This paradox suggests that extreme tolerance could result in intolerance towards those who threaten tolerance. It highlights a critical dilemma in maintaining an open and accepting society.

Voting Paradox:[93] In the realm of social choice theory, Arrow's impossibility theorem underlines the complexities involved in aggregating individual preferences into a collective decision. It asserts that no voting system can satisfy a set of rational criteria simultaneously, including unanimity, absence of dictatorial influence, and transitivity.

Psychology :

Dunning-Kruger Effect :[94] Incompetent individuals often tend to overestimate their abilities due to cognitive bias.

Paradox of Choice :[95] Too many choices can lead to dissatisfaction and anxiety despite the perceived benefit of having choices.

Philosophy:

Ship of Theseus:[96] This philosophical paradox presents a thought experiment where each part of a ship is gradually replaced until none of the original components remain. This raises the question: Is the fully restored ship still the same as the original? This thought experiment explores deep philosophical questions about the identity of objects and the nature of their essence.

Conclusion and Future Research Directions

These examples from various fields highlight how paradoxes challenge conventional wisdom, sparking deeper exploration and presenting fresh perspectives. Among various academic disciplines, there are points we should consider regarding artificial intelligence, or AI, which has been receiving significant attention in recent.

Recently, the issue of artificial intelligence machines has become a focal point of discussion among scientists and philosophers. As machines with AI continue to evolve through deep learning, a fundamental question arises : Can we attribute the capability of thinking to these machines?

Studying paradoxes serves several important purposes across various disciplines and aspects of human comprehension. In essence, studying paradoxes helps us grow intellectually, uncover hidden truths, hones our thinking, and contribute to the advancement of knowledge in various fields. Paradoxes challenge us to question, explore, and think critically, fostering a deeper and more intricate understanding of the world around us.

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